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Free Energy of the Solvable Chiral Potts Model

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Very recently, it has been shown that there are chiral N-state Potts models in statistical mechanics that satisfy the star-triangle relation. Here it is shown that the relation implies that the free energy (and its derivatives) satisfies certain functional relations. These can be used to obtain the free energy: in particular, we expand about the critical case and find that the exponent α is $1-2/N$.

KEY WORDS: Statistical mechanics; lattice statistics; Potts model.

1. INTRODUCTION

The star-triangle (or Yang–Baxter) relation and its generalizations play a central role in the theory of exactly solvable models in statistical mechanics.[(1,2)] Very recently,[(3–5)] solutions of this relation have been found for a restricted class of N-state chiral Potts models. (I shall call models "solvable" if they belong to this class.) Unlike the previous solutions for other models, they do not have the "difference property," where the Boltzmann weights of a vertex can be expressed as a function of the difference of the "rapidities" of the two lines through that vertex.

For the other models, this difference property makes it straightforward to obtain the free energy from the inversion relation (or unitarity condition)[(2,6,7)] and to obtain single-spin expectation values (e.g., the Ising model magnetization) by using the corner transfer matrix technique.[(2,8)] Without it, it is not clear how best to proceed.

Here I adapt the "399th" method used for the Ising model (§11.7 of ref. 2; ref. 9). I show that the star-triangle relation implies certain functional relations for the free energy and correlations of the solvable chiral Potts

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model and that these can be solved. In particular, I expand about the critical case, obtaining the result

$$\alpha = 1 - 2/N \tag{1.1}$$

for the critical exponent α.

2. GENERAL Z-INVARIANT MODEL

First consider a very general Z-invariant model.[10,11] One has a collection of straight lines in the plane, line j carrying a rapidity variable p_j. These lines form a graph $\mathscr{G}$, spins live on alternate faces of $\mathscr{G}$, and two spins are adjacent if their faces touch at a vertex of $\mathscr{G}$. Each spin σ takes the values $1,\ldots, N$. Two adjacent spins a and b contribute to the partition function Z a weight $W_{pq}(a, b)$ if they are arranged as in Fig. 1a (in which case we call them a W-pair) and a weight $\bar{W}_{pq}(a, b)$ if arranged as in Fig. 1b (a $\bar{W}$-pair). The p and q are the rapidities of the two intervening lines; $W_{pq}(b, a)$ is not necessarily equal to $W_{pq}(a, b)$. Thus,

$$Z = \sum_{\{\sigma\}} \prod_{\langle ij \rangle} W_{pq}(\sigma_i, \sigma_j) \tag{2.1}$$

where the sum is over all values of all the spins $\sigma_1, \sigma_2,\ldots$; the product is over all adjacent pairs (i, j) of spins σ_i, σ_j. For each pair or "edge," σ_i and σ_j must be ordered as are a and b in Fig. 1 (p and q being the intervening line rapidities), and W_{pq} must be replaced by $\bar{W}_{pq}$, as they are a $\bar{W}$-pair.

A model is "Z-invariant" if W, $\bar{W}$ satisfy the star-triangle relation:

$$\sum_{d=1}^{N} \bar{W}_{qr}(b, d)\, W_{pr}(a, d)\, \bar{W}_{pq}(d, c) = R_{pqr}\, W_{pq}(a, b)\, \bar{W}_{pr}(b, c)\, W_{qr}(a, c) \tag{2.2}$$

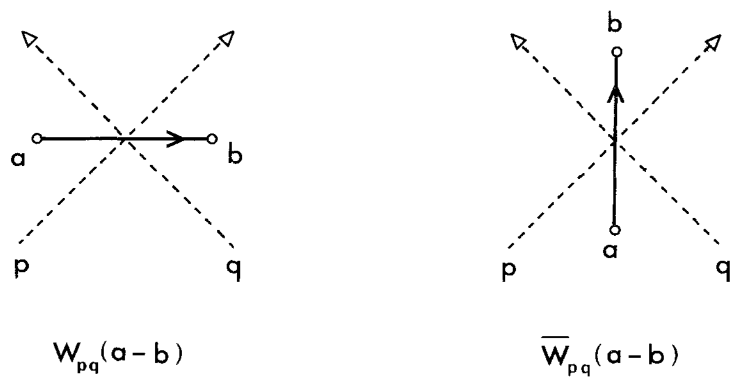

Fig. 1. The two Boltzmann weights depending on the orientation of the spin pair with respect to the rapidity lines p and q.

which can be represented graphically as in Fig. 2 of ref. 5. (A more symmetric form of the equation can be obtained by reversing the direction of line q.)

2.1. Expectation Values

I begin with some general comments that apply to any model satisfying (2.2), e.g., the Ising, self-dual Ashkin–Teller (which is equivalent to the eight-vertex model), or critical Potts model.[2,11] The product in (2.1) is a function of $\sigma_1, \sigma_2, \ldots$: write it as $P\{\sigma\}$. Let $F(a, b)$ be an arbitrary function of two adjacent spins a and b (and other parameters): then its expectation value on an edge (i, j), with weight function $W_{pq}(\sigma_i, \sigma_j)$, is

$$\langle F \mid W_{pq} \rangle = Z^{-1} \sum_{\{\sigma\}} F(\sigma_i, \sigma_j)\, P\{\sigma\} \tag{2.3}$$

On an edge with weight function $\bar{W}_{pq}(\sigma_i, \sigma_j)$, its expectation value is still given by (2.3), but is written as $\langle F \mid \bar{W}_{pq} \rangle$. A vital property of a Z-invariant model is that in the large-lattice limit, $\langle F \mid W_{pq} \rangle$ and $\langle F \mid \bar{W}_{pq} \rangle$ depend on the rapidities p and q of the two lines between spins σ_i and σ_j, *but not on the rapidities of any other lines.*[10,11]

Derivatives of $\ln Z$ can be expressed as sums-over-edges of particular expectation values, notably

$$\begin{aligned} A_{pq} &= \left\langle \frac{\partial}{\partial p} \ln W_{pq} \,\middle|\, W_{pq} \right\rangle, \qquad B_{pq} = \left\langle \frac{\partial}{\partial q} \ln W_{pq} \,\middle|\, W_{pq} \right\rangle \\ \bar{A}_{pq} &= \left\langle \frac{\partial}{\partial p} \ln \bar{W}_{pq} \,\middle|\, \bar{W}_{pq} \right\rangle, \qquad \bar{B}_{pq} = \left\langle \frac{\partial}{\partial q} \ln \bar{W}_{pq} \,\middle|\, \bar{W}_{pq} \right\rangle \end{aligned} \tag{2.4}$$

For all the models mentioned above except the critical Potts model, the weights W_{pq}, $\bar{W}_{pq}$ of all edges also depend on a single "universal" modulus k, so for these we also need

$$C_{pq} = \left\langle \frac{\partial}{\partial k} \ln W_{pq} \,\middle|\, W_{pq} \right\rangle, \qquad \bar{C}_{pq} = \left\langle \frac{\partial}{\partial k} \ln \bar{W}_{pq} \,\middle|\, \bar{W}_{pq} \right\rangle \tag{2.5}$$

These functions $A_{pq}, \ldots, \bar{C}_{pq}$ depend only on the rapidities p and q (and possibly the modulus k): they are the same for any lattice.

2.2. Honeycomb and Triangular Lattices

Now take $\mathscr{G}$ to be the Kagomé lattice. The spins form either the honeycomb or the triangular lattice, depending on which set of alternating faces of $\mathscr{G}$ they occupy (Chapter 11 of ref. 2). Give parallel lines equal rapidity, so there are three distinct rapidities p, q, and r, as in Fig. 2 of ref. 5. Let the honeycomb (triangular) lattice have $2L$ (L) sites, and let $Z_H(p, q, r)$ [$Z_T(p, q, r)$] be its partition function. Then from (2.2)

$$Z_H(p, q, r) = R_{pqr}^L Z_T(p, q, r) \tag{2.6}$$

The honeycomb (triangular) lattice has L edges with weight function $\bar{W}_{qr}$ (W_{qr}), L with W_{pr} ($\bar{W}_{pr}$), and L with $\bar{W}_{pq}$ (W_{pq}). Differentiating $\ln Z_H(p, q, r)$, using (2.1) and (2.4), we obtain

$$\begin{aligned} L^{-1} \frac{\partial}{\partial p} \ln Z_H(p, q, r) &= A_{pr} + \bar{A}_{pq} \\ L^{-1} \frac{\partial}{\partial q} \ln Z_H(p, q, r) &= \bar{A}_{qr} + \bar{B}_{pq} \\ L^{-1} \frac{\partial}{\partial r} \ln Z_H(p, q, r) &= \bar{B}_{qr} + B_{pr} \end{aligned} \tag{2.7}$$

I emphasize that any rapidity dependence is here shown explicitly: e.g., A_{pr} is independent of q.

It follows from (2.7) that

$$\frac{\partial}{\partial q} \bar{A}_{pq} = \frac{\partial}{\partial p} \bar{B}_{pq} \tag{2.8}$$

and hence $\exists$ a function $\bar{\psi}_{pq}$ such that

$$\bar{A}_{pq} = -\frac{\partial}{\partial p} \bar{\psi}_{pq}, \qquad \bar{B}_{pq} = -\frac{\partial}{\partial q} \bar{\psi}_{pq} \tag{2.9}$$

Similarly, $\exists \psi_{pq}$ such that

$$A_{pq} = -\frac{\partial}{\partial p} \psi_{pq}, \qquad B_{pq} = -\frac{\partial}{\partial q} \psi_{pq} \tag{2.10}$$

It follows immediately from (2.7) that $L^{-1} \ln Z_H(p, q, r) + \bar{\psi}_{qr} + \psi_{pr} + \bar{\psi}_{pq}$ is a "constant" (i.e., independent of p, q, and r). A similar argument (with barred and unbarred variables interchanged) applies for the triangular lattice. The two constants can be absorbed into ψ_{pq} and $\bar{\psi}_{pq}$ (which are then defined uniquely), giving

$$\begin{aligned} L^{-1} \ln Z_H(p, q, r) &= -\bar{\psi}_{qr} - \psi_{pr} - \bar{\psi}_{pq} \\ L^{-1} \ln Z_T(p, q, r) &= -\psi_{qr} - \bar{\psi}_{pr} - \psi_{pq} \end{aligned} \tag{2.11}$$

From (2.6), it follows that

$$R_{pqr} = f_{pq} f_{qr}/f_{pr} \tag{2.12}$$

where

$$\ln f_{pq} = \psi_{pq} - \bar{\psi}_{pq} + g_p - g_q \tag{2.13}$$

where g_p is an arbitrary function. This verifies the conjecture (12) of ref. 5.

We can think of ψ_{pq}, $\bar{\psi}_{pq}$ as free energies per edge (one for each edge type). Usually, if we can solve the star-triangle relation (2.2), then we know the factor R_{pqr}, and hence f_{pq}. Then (2.13) is a relation between the two free energies.

In all the models, we can ensure that

$$W_{pp}(a, b) = 1, \qquad \lim_{q \to p} \bar{W}_{pq}(a, b)/\bar{W}_{pq}(0, 0) = \delta_{a,b} \tag{2.14}$$

Taking $p = q = r$, the honeycomb and triangular models become trivial, and from (2.11) we find

$$\psi_{pp} = 0, \qquad \lim_{q \to p} [\bar{\psi}_{pq} + \ln \bar{W}_{pq}(0, 0)] = 0 \tag{2.15}$$

2.3. Square Lattice

If we let $r \to q$ in the honeycomb and triangular models, the W_{qr} edges disappear, while $\bar{W}_{qr}$ edges contract to a point. Both models then reduce to the square-lattice chiral Potts model of L sites, with weight functions W_{pq} and $\bar{W}_{pq}$ on horizontal and vertical edges, respectively. Hence, using (2.15), the square lattice partition function is given by

$$L^{-1} \ln Z_{\mathrm{Sq}}(p, q) = -\psi_{pq} - \bar{\psi}_{pq} \tag{2.16}$$

2.4. *k*-Derivatives

If the model also has a variable modulus k on which the weights, expectations values, and free energies implicitly depend, then differentiating the partition functions, using (2.11), gives

$$\begin{aligned} \bar{C}_{pq} &= -\frac{\partial}{\partial k} \bar{\psi}_{pq} + h_p - h_q \\ C_{pq} &= -\frac{\partial}{\partial k} \psi_{pq} + h_q - h_p \end{aligned} \tag{2.17}$$

where the function h_p is yet to be determined.

To summarize: ψ_{pq} and $\bar{\psi}_{pq}$ are uniquely defined by (2.9)–(2.11) and can be regarded as "edge free energies." Their difference is given by (2.13), and their derivatives (2.9), (2.10), and (2.17) yield the Z-invariant edge-expectation functions $A_{pq},..., \bar{C}_{pq}$. The single-rapidity functions g_p and h_p also have to be determined from these equations. All these functions may depend implicitly on the elliptic modulus k (and on any other "universal" variables, e.g., the four-spin coupling in the eight-vertex/Ashkin–Teller model).

If one has some further information, notably linear relations between A_{pq}, B_{pq}, C_{pq} (and $\bar{A}_{pq}$, $\bar{B}_{pq}$, $\bar{C}_{pq}$), then it may be possible to solve the system of equations for ψ_{pq}, $\bar{\psi}_{pq}$. Basically this is the method used in ref. 9 to solve the Ising model. Here we adapt it to the chiral Potts model.

3. CHIRAL POTTS MODEL: EQUATIONS FOR ψ_{pq}, $\bar{\psi}_{pq}$

Now we specialize to the chiral Potts model. The functions W_{pq}, $\bar{W}_{pq}$ are defined in ref. 5. Here it is convenient to work not with the $a_p,..., d_p$ therein, but with variables θ_p, ϕ_p, u_p, v_p defined by

$$\begin{gathered} e^{i\theta_p} = e^{-\pi i/N} b_p/c_p, \qquad e^{i\phi_p} = a_p/d_p \\ u_p = N(\theta_p + \phi_p)/2, \qquad v_p = N(\theta_p - \phi_p)/2 \end{gathered} \tag{3.1}$$

From Eq. (9) of ref. 5

$$\begin{gathered} \sin v_p = k \sin u_p \\ \cos v_p = (1 - k^2 \sin^2 u_p)^{1/2} \end{gathered} \tag{3.2}$$

Hence if k is given, then any one of θ_p, ϕ_p, u_p, v_p specifies the other three. They are all functions of the rapidity p, which we can choose to be u_p, i.e.,

$$u_p \equiv p \tag{3.3}$$

We shall need two functions:

$$\begin{gathered} s_p = \partial v_p/\partial k = \sin u_p/\cos v_p \\ t_p = \partial v_p/\partial p = k \cos u_p/\cos v_p \end{gathered} \tag{3.4}$$

We shall usually take k, u_p, v_p to be real, with

$$0 < k < 1, \qquad -\pi/2 < v_p < \pi/2 \tag{3.5}$$

Define a function $T(\theta; n)$, for integer n, by

$$\frac{T(\theta; n)}{T(\theta; 0)} = \left(\cos\frac{N\theta}{2}\right)^{-n/N} \prod_{j=1}^{n} \sin\left[-\frac{\theta}{2} + \frac{\pi(2j-1)}{2N}\right]; \qquad \prod_{n=0}^{N-1} T(\theta; n) = 1 \tag{3.6}$$

Then the Boltzmann weight functions of the chiral Potts model are

$$W_{pq}(a, b) = T(\theta_q - \phi_p; a-b)/T(\theta_p - \phi_q; a-b)$$
$$\bar{W}_{pq}(a, b) = T\left(\phi_p - \phi_q + \frac{\pi}{N}; a-b\right) \Big/ T\left(\theta_q - \theta_p - \frac{\pi}{N}; a-b\right) \tag{3.7}$$

They (and the constituent T factors) are positive if $0 < u_q - u_p < \pi$ [this follows from (3.33)]; they are periodic functions of $a-b$ with period N and are normalized so that

$$\prod_{n=0}^{N-1} W_{pq}(b+n, b) = \prod_{n=0}^{N-1} \bar{W}_{pq}(b+n, b) = 1 \tag{3.8}$$

They satisfy the constraints (2.14).

Obviously $W_{pq}(a, b)$ depends on p, q, and k only via $\theta_q - \phi_p$ and $\theta_p - \phi_q$, i.e., via $u_q - u_p$ and $v_p + v_q$. If we temporarily regard $W_{pq}(a, b)$ as a function of $u_p - u_q$, $v_p + v_q$, and $v_q - v_p$ (instead of p, q, and k), then its derivative with respect to $v_q - v_p$ vanishes. Returning to the variables p, q, and k, this implies

$$\mathscr{L}_{pq} W_{pq}(a, b) = 0 \tag{3.9}$$

where

$$\mathscr{L}_{pq} = (s_p + s_q)\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right) - (t_p + t_q)\frac{\partial}{\partial k} \tag{3.10}$$

Hence, using (4), $\langle \mathscr{L}_{pq} \ln W_{pq} \mid W_{pq} \rangle = \langle 0 \mid W_{pq} \rangle = 0$. On the other hand, from (2.4) and (2.5), and (2.9), (2.10), and (2.17):

$$\begin{aligned} \langle \mathscr{L}_{pq} \ln W_{pq} \mid W_{pq} \rangle &= (s_p + s_q)(A_{pq} + B_{pq}) - (t_p + t_q)\, C_{pq} \\ &= -\mathscr{L}_{pq} \psi_{pq} - (t_p + t_q)(h_q - h_p) \end{aligned} \tag{3.11}$$

so

$$\mathscr{L}_{pq} \psi_{pq} = (t_p + t_q)(h_p - h_q) \tag{3.12}$$

Similarly, $\bar{W}_{pq}(a, b)$ depends only on $u_q - u_p$ and $v_q - v_p$, so its derivative with respect to $v_p + v_q$ vanishes, giving

$$\bar{\mathscr{L}}_{pq} \bar{W}_{pq}(a, b) = 0 \tag{3.13}$$

where

$$\bar{\mathscr{L}}_{pq} = (s_p - s_q)\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right) - (t_p - t_q)\frac{\partial}{\partial k} \tag{3.14}$$

This leads to

$$\bar{\mathscr{L}}_{pq}\bar{\psi}_{pq} = (t_q - t_p)(h_p - h_q) \tag{3.15}$$

Eliminating $\bar{\psi}_{pq}$ between (2.13) and (3.15),

$$\bar{\mathscr{L}}_{pq}(\psi_{pq} - \ln f_{pq} + g_p - g_q) = (t_q - t_p)(h_p - h_q) \tag{3.16}$$

The rest of this paper is concerned with solving the two equations (3.12) and (3.16) for g_p, h_p, and ψ_{pq}.

3.1. Consistency Condition

In terms of variables $u_q - u_p$, $v_q - v_p$, and $v_q + v_p$, the operators $\mathscr{L}_{pq}$ and $\bar{\mathscr{L}}_{pq}$ are the derivatives with respect to $v_q - v_p$ and $v_q + v_p$, respectively, multiplied on the left by

$$2(s_p t_q - s_q t_p) = 2k \sin(u_p - u_q)/(\cos v_p \cos v_q)$$

Defining a function

$$\gamma_{pq} = k^{-1} \cos v_p \cos v_q \tag{3.17}$$

it follows (because of the general mathematical relation $\partial^2/\partial x\, \partial y = \partial^2/\partial y\, \partial x$) that $\mathscr{L}_{pq}$ and $\bar{\mathscr{L}}_{pq}$ satisfy the commutation relation

$$\bar{\mathscr{L}}_{pq}\gamma_{pq}\mathscr{L}_{pq} = \mathscr{L}_{pq}\gamma_{pq}\bar{\mathscr{L}}_{pq} \tag{3.18}$$

This can be verified directly.

Multiplying (3.12) on the left by $\bar{\mathscr{L}}_{pq}\gamma_{pq}$, (3.16) by $\mathscr{L}_{pq}\gamma_{pq}$, and subtracting, it follows that the single-rapidity functions g_p and h_p must satisfy the consistency condition:

$$\begin{aligned}\bar{\mathscr{L}}_{pq}\gamma_{pq}&(t_p + t_q)(h_p - h_q)\\ &= \mathscr{L}_{pq}\gamma_{pq}\{(t_q - t_p)(h_p - h_q) + \bar{\mathscr{L}}_{pq}(\ln f_{pq} - g_p + g_q)\}\end{aligned} \tag{3.19}$$

3.2. The function f_{pq}

We need to know f_{pq} (or at least $\bar{\mathscr{L}}_{pq} \ln f_{pq}$). This is given in Eq. (13) of ref. 5. Unfortunately, this equation is only a conjecture, but it has been stringently tested numerically, is correct for the $k=0$ and $N=2$ cases, and has the right symmetry properties. It would be amazing if it were wrong.

Equation (13) of ref. 5 can be simplified by noting that the numerator is the Nth root of the determinant of an N by N matrix with elements

$M_{ab} = \bar{W}_{pq}(a, b)$. This matrix depends on the variables p and q (and k), so we can write it as $\mathbf{M}_{pq}$. Then it satisfies the inversion relation

$$\mathbf{M}_{pq}\mathbf{M}_{qp} = S_{pq}\mathbb{1} \tag{3.20}$$

where

$$S_{pq} = N\frac{\sin(u_q - u_p)}{\sin[(u_q - u_p)/N]}\left\{4\sin\frac{N(\theta_q - \theta_p)}{2}\sin\frac{N(\phi_q - \phi_p)}{2}\right\}^{(1-N)/N} \tag{3.21}$$

Because of the normalization (3.8), the denominator in Eq. (13) of ref. 5 is unity. It follows that $f_{pq}f_{qp} = S_{pq}$. Noting that the elements of M_{pq} are rational functions of $\exp[i(\theta_q - \theta_p)]$ and $\exp[i(\phi_q - \phi_p)]$, one can go on to establish that

$$\begin{aligned} f_{pq}^N &= \det \mathbf{M}_{pq} \\ &= \frac{N^{N/2}\prod_{j=1}^{N-1}\{2\sin[(u_q - u_p + \pi j)/N\}^j}{\{4\sin[N(\theta_q - \theta_p)/2]\sin[N(\phi_q - \phi_p)/2]\}^{(N-1)/2}} \end{aligned} \tag{3.22}$$

Thus f_{pq} depends on p, q, and k only via $u_q - u_p$ and $v_q - v_p$. It follows at once that

$$\bar{\mathscr{L}}_{pq}\ln f_{pq} = 0 \tag{3.23}$$

Indeed, this property follows at once from (3.13), provided one simply notes that f_{pq} depends on p, q, and k only via $\bar{W}_{pq}(a, b)$. It is the main property of f_{pq} that we shall use herein.

3.3. Low-Temperature Limit: $k \to 1$

One can easily see that (3.12) and (3.16) are unchanged by adding to ψ_{pq} an arbitrary function of $u_q - u_p$ only. To fix ψ_{pq}, we therefore need an extra piece of information, and this can be obtained by looking at a "low-temperature"-type limit, where $k \to 1$, while u_p, u_q remain fixed, in the range $-\pi/2 < u_p$, $u_q < \pi/2$. Then ϕ_p, $\phi_q \to 0$ and $-\pi/N < \theta_p$, $\theta_q < \pi/N$; $\ln W_{pq}(a, b)$ tends to a finite limit, with $W_{pq}(0, 0) \geqslant W_{pq}(a, b)$, while $\bar{W}_{pq}(a, b)/\bar{W}_{pq}(0, 0) \to \delta_{a,b}$.

The system is then ordered with all spins equal, so

$$\psi_{pq} = -\ln W_{pq}(0, 0), \qquad \bar{\psi}_{pq} + \ln \bar{W}_{pq}(0, 0) = \bar{\psi}_{pq} + \ln f_{pq} = 0 \tag{3.24}$$

Hence, using (3.7),

$$\psi_{pq} = g_q - g_p \tag{3.25a}$$

where

$$g_p = \sum_{j=1}^{N} \frac{N+1-2j}{N} \ln \left\{ 2 \sin \frac{(j-1/2)\pi - u_p}{N} \right\} \tag{3.25b}$$

We see at once that (2.13) is satisfied, so g_p therein is given by (3.25b), provided $k=1$ and $-\pi/2 < u_p < \pi/2$.

3.4. Symmetries

In ref. 5 it was remarked that the model has a rotation symmetry: $p, q \to q, Rp$, and a reflection symmetry: $p, q \to Sq, Sp$. In terms of our variables, Rp and Sp are defined by

$$\begin{aligned} &\theta_{Rp} = \phi_p + \pi/N, && \phi_{Rp} = \theta_p + \pi/N, \\ &u_{Rp} = u_p + \pi, && v_{Rp} = -v_p \\ &\theta_{Sp} = -\phi_p - \pi/N, && \phi_{Sp} = -\theta_p - \pi/N \\ &u_{Sp} = -u_p - \pi, && v_{Sp} = v_p \end{aligned} \tag{3.26}$$

(Thus, $Rp = p + \pi$, $Sp = -p - \pi$.) Then

$$W_{q,Rp}(a,b) = \bar{W}_{pq}(a,b), \qquad \bar{W}_{q,Rp}(a,b) = W_{pq}(b,a) \tag{3.27}$$

$$W_{Sq,Sp}(a,b) = W_{p,q}(a,b), \qquad \bar{W}_{Sq,Sp}(a,b) = \bar{W}_{p,q}(b,a) \tag{3.28}$$

It follows that

$$\begin{aligned} &A_{q,Rp} = \bar{B}_{pq}, && B_{q,Rp} = \bar{A}_{pq}, && C_{q,Rp} = \bar{C}_{pq} \\ &A_{Sq,Sp} = -B_{pq}, && B_{Sq,Sp} = -A_{pq}, && C_{Sq,Sp} = C_{pq} \end{aligned} \tag{3.29}$$

and similarly with barred and unbarred variables interchanged. Also, replacing p, q, and r by q, r, and Rp (and by Sr, Sq, and Sp) leaves the honeycomb and triangular lattices unchanged, apart from a rotation (reflection). From (2.9)–(2.11) and (2.17), we get

$$\begin{aligned} &\psi_{q,Rp} = \bar{\psi}_{pq} + \eta, && \bar{\psi}_{q,Rp} = \psi_{pq} - \eta \\ &\psi_{Sq,Sp} = \psi_{pq}, && \bar{\psi}_{Sq,Sp} = \bar{\psi}_{pq} \end{aligned} \tag{3.30}$$

$$h_{Rp} - h_p = d\eta/dk \tag{3.31}$$

$$h_{Sp} + h_p = \zeta \tag{3.32}$$

where $\eta \equiv \eta(k)$, $\zeta \equiv \zeta(k)$ are functions only of k.

From (3.1) and (3.2), we can establish the rather remarkable identity

$$4\sin[N(\theta_q-\theta_p)/2]\sin[N(\phi_q-\phi_p)/2]\cos[N(\theta_p-\phi_q)/2] \times\cos[N(\phi_p-\theta_q)/2]=k'^2\sin^2(u_q-u_p) \tag{3.33}$$

where

$$k'=(1-k^2)^{1/2} \tag{3.34}$$

is the conjugate modulus of k. Using this and (3.22), we find

$$f_{pq}f_{q,Rp}=N/k'^{(N-1)/N} \tag{3.35}$$

and $f_{Sq,Sp}=f_{pq}$. From (2.13) and (3.30), it follows that

$$\begin{aligned} g_{Rp}-g_p&=2\eta-\ln N+\frac{N-1}{N}\ln k' \\ g_{Sp}+g_p&=\tau \end{aligned} \tag{3.36}$$

where $\tau\equiv\tau(k)$ is a function only of k.

3.5. Inversion Relation

Another symmetry is provided by (3.20), together with

$$W_{pq}(a,b)\,W_{qp}(a,b)=1 \tag{3.37}$$

This implies that if we consider a transfer matrix going in the direction of the $\bar{W}$ edges, for both the honey comb and triangular lattices this matrix is inverted by interchanging p with r (apart from scalar factors involving S_{pq}, S_{qr}, S_{pr}). This negates the free energy, so from (2.11) we get

$$\bar{\psi}_{qr}+\psi_{pr}+\bar{\psi}_{pq}+\bar{\psi}_{qp}+\psi_{rp}+\bar{\psi}_{rq}=-\ln(S_{pq}S_{qr}) \tag{3.38}$$

$$\psi_{qr}+\bar{\psi}_{pr}+\psi_{pq}+\bar{\psi}_{qr}+\psi_{pr}+\bar{\psi}_{pq}=-\ln S_{pr} \tag{3.39}$$

From these equations, together with $S_{pq}=S_{qp}$, we can deduce that

$$\begin{aligned} \psi_{pq}+\psi_{qp}&=0 \\ \bar{\psi}_{pq}+\bar{\psi}_{qp}&=-\ln S_{pq}=-\ln f_{pq}f_{qp} \end{aligned} \tag{3.40}$$

These relations are consistent with (2.13).

For the eight-vertex and other previously solved models, ψ_{pq} is a function only of $p-q$, and there is the very direct "inversion relation method" for obtaining ψ_{pq} immediately from (3.40).[7,12] However, this method does require an assumption that ψ_{pq} be analytic in a particular complex domain. Here I use this method only to obtain the isolated $k=0$ result (6.15) (previously obtained by Fateev and Zamolodchikov),[15] and in principle one can avoid doing even this. For nonzero k I emphasize that here I do *not* use this method, and do *not* make such an analyticity assumption (partly because sufficient understanding is lacking to be able to do so with any confidence). My only assumptions (some of which can probably be easily proved) are:

1. Local differentiability for u_p, u_q, k real, $0<u_q-u_p<\pi$, $0<k<1$.
2. For $-\pi/2<u_p<u_q<\pi/2$, that $A_{pq},\ldots, \bar{C}_{pq}$ are Taylor-expandable in powers of $k'^2=1-k^2$ (this fits with series expansions and the $N=2$ Ising case).
3. That ψ_{pq}, $A_{pq},\ldots, \bar{C}_{pq}$ tend to finite limits as $k\to 0$.

3.6. Derivatives of the Square Lattice Free Energy

From (2.16), the free energy per site of the square lattice is

$$\psi_{pq}^{(\mathrm{Sq})}=\psi_{pq}+\bar{\psi}_{pq}=2\psi_{pq}+g_p-g_q-\ln f_{pq} \tag{3.41}$$

Let us define, as functions of p, q, and k,

$$\Lambda_{pq}=\psi_{pq}^{(\mathrm{Sq})}+\ln f_{pq}=2\psi_{pq}+g_p-g_q \tag{3.42}$$

$$c_{pq}=\cos u_p \cos u_q-k'^2 \sin u_p \sin u_q \tag{3.43}$$

$$x_p=\partial g_p/\partial p, \qquad y_p=2kh_p-k\,\partial g_p/\partial k \tag{3.44}$$

Then from (3.10), (3.12), (3.14), (3.16), and (3.23) it follows that

$$\frac{\partial \Lambda_{pq}}{\partial k}=\frac{\sin(u_p+u_q)\,(x_q-x_p)+c_{pq}(y_q-y_p)}{k\cos v_p \cos v_q} \tag{3.45}$$

$$\left(\frac{\partial}{\partial p}+\frac{\partial}{\partial q}\right)\Lambda_{pq}=\frac{c_{pq}(x_q-x_p)-k'^2\sin(u_p+u_q)\,(y_q-y_p)}{\cos v_p\cos v_q} \tag{3.46}$$

Thus if we know the single-rapidity functions x_p, y_p, then we can obtain Λ_{pq} by integrating either (3.45) or (3.46). The consistency condition (3.19) ensures that both ways can give the same result.

In the next two sections I show that this condition defines x_p, y_p (to within additive terms independent of p).

4. EQUATIONS FOR x_p, y_p

Define

$$\mathscr{L}_p = s_p\left(\frac{\partial}{\partial p}+\frac{\partial}{\partial q}\right) - t_p\,\frac{\partial}{\partial k} \tag{4.1}$$

Then from (3.10) and (3.14),

$$\mathscr{L}_{pq} = \mathscr{L}_p + \mathscr{L}_q, \qquad \bar{\mathscr{L}}_{pq} = \mathscr{L}_p - \mathscr{L}_q \tag{4.2}$$

while from (3.18) $\mathscr{L}_p \gamma_{pq} \mathscr{L}_q = \mathscr{L}_q \gamma_{pq} \mathscr{L}_p$. Using (3.23), it follows that (3.19) can be written as

$$D_{pq} + D_{qp} = 0 \tag{4.3}$$

where

$$D_{pq} = \mathscr{L}_q \gamma_{pq}(2t_q h_p + \mathscr{L}_q\, g_p) - \mathscr{L}_p \gamma_{pq}(2t_p h_p + \mathscr{L}_p\, g_p) \tag{4.4}$$

From now on we shall find it convenient to work with the functions x_p, y_p defined by (3.44), rather than with g_p, h_p. Using (3.4) and (3.17), we can write (4.4) as

$$\begin{aligned} D_{pq} = {} & \mathscr{L}_q[k^{-1}\cos v_p\,(x_p \sin u_q + y_p \cos u_q)] \\ & - \mathscr{L}_p[k^{-1}\cos v_q\,(x_p \sin u_p + y_p \cos u_p)] \end{aligned} \tag{4.5}$$

Define two auxiliary functions

$$\begin{aligned} \hat{x}_p &= \frac{\partial x_p}{\partial p} + kk'^2\,\frac{\partial y_p}{\partial k} - 2y_p \\ \hat{y}_p &= \frac{\partial y_p}{\partial p} - k\,\frac{\partial x_p}{\partial k} \end{aligned} \tag{4.6}$$

Then, on expanding (4.5), using (3.4) and (3.2), we find that

$$D_{pq} = \sin(u_q - u_p) E_{pq}/(k\cos v_p \cos v_q) \tag{4.7}$$

where

$$\begin{aligned} E_{pq} = {} & \hat{x}_p \sin(u_p + u_q) + 2x_p \cos(u_p + u_q) \\ & + \hat{y}_p(\cos u_p \cos u_q - k'^2 \sin u_p \sin u_q) \end{aligned} \tag{4.8}$$

If we now define

$$X_p = \hat{x}_p \sin u_p + (2x_p + \hat{y}_p)\cos u_p$$
$$Y_p = \hat{x}_p \cos u_p - (2x_p + k'^2\hat{y}_p)\sin u_p \tag{4.9}$$

then

$$E_{pq} = X_p \cos u_q + Y_p \sin u_q \tag{4.10}$$

The consistency condition (4.2) is equivalent to $E_{pq} = E_{qp}$ for all p, q. From (4.10) it follows that there are only two independent linear combinations of the functions X_p, Y_p, $\cos u_p$, $\sin u_p$. Hence there must exist parameters α, β, δ (independent of p, but still implicitly dependent on k) such that

$$X_p = \alpha \cos u_p + \beta \sin u_p, \qquad Y_p = \beta \cos u_p + \delta \sin u_p \tag{4.11}$$

[The equality of the two parameters β is a consequence of (4.10).]

Solving (4.9) and (4.11) for $\hat{x}_p$, $\hat{y}_p$, then using (4.6), we get

$$2x_p + \left(\frac{\partial y_p}{\partial p} - k\frac{\partial x_p}{\partial k}\right)\cos^2 v_p = \alpha - (\alpha+\delta)\sin^2 u_p \tag{4.12a}$$

$$-k^2 x_p \sin 2u_p + \left(\frac{\partial x_p}{\partial p} + kk'^2\frac{\partial y_p}{\partial k} - 2y_p\right)\cos^2 v_p$$
$$= \frac{1}{2}(k'^2\alpha + \delta)\sin 2u_p + \beta\cos^2 v_p \tag{4.12b}$$

This is a pair of coupled linear partial differential equations for x_p, y_p. We can eliminate y_p by dividing (4.12b) by $\cos^2 v_p$, differentiating with respect to p, and using (4.12a). The result simplifies considerably if we introduce a function G_p by

$$x_p = k^2 G_p/\cos v_p \tag{4.13}$$

Then we obtain

$$k\frac{\partial}{\partial k}\left(kk'^2\frac{\partial G_p}{\partial k}\right) - k^2 G_p + \frac{\partial^2 G_p}{\partial p^2} = \frac{k^2(\lambda + \mu\cos 2u_p)}{\cos v_p} \tag{4.14}$$

where

$$\lambda = \frac{\alpha - k'^2\delta}{k^4} + \frac{k'^2}{2k^3}\frac{d}{dk}(\delta - \alpha), \qquad \mu = -\frac{k}{2}\frac{d}{dk}\left[\frac{k'^2}{k^4}(\alpha+\delta)\right] \tag{4.15}$$

Choice of Arbitrary Parameters

The equations (4.12) are homogeneous and linear in the unknown functions x_p, y_p, α, β, δ. They admit two particularly simple solutions: Case (i)

$$\begin{aligned} &x_p = \alpha = \delta = 0 \\ &y_p = \text{independent of } p \\ &\beta = kk'^2 \frac{dy_p}{dk} - 2y_p \end{aligned} \tag{4.16}$$

Case(ii)

$$\begin{aligned} &y_p = \beta = 0 \\ &x_p = \text{independent of } p \\ &\alpha = 2x_p - k\, dx_p/dk \\ &\delta = -2x_p + kk'^2\, dx_p/dk \end{aligned} \tag{4.17}$$

We are therefore free to add such solutions, provided we leave unchanged (as we can) the $k = 1$ boundary condition (3.25) and the $k \to 0$ finiteness requirements.

From (3.44), adding a solution of type (i) merely increments g_p and h_p by terms independent of p. This has no effect on (2.13), (3.12), or (3.16), and so it does not change the free energies ψ_{pq}, $\bar{\psi}_{pq}$. It does change β, so we can use this freedom to ensure that

$$\beta = 0 \tag{4.18}$$

Adding a solution of type (ii) increments x_p by some arbitrary function $\phi(k)$ (independent of p). This induces increments $p\phi(k)$ and $p\phi'(k)/2$ in g_p and h_p and causes both ψ_{pq} and $-\bar{\psi}_{pq}$ to be incremented by $(q-p)\,\phi(k)/2$. This affects (in a trivial way) the expressions (2.9) and (2.10) for the expectation values $A_{pq},\ldots, \bar{B}_{pq}$. However, *it does not change* the expressions (2.11) and (2.16) for the free energies of the honeycomb, triangular, and square lattices; nor does it change (3.45) or (3.46). Thus, if we only consider these total free energies, we are free to add a solution of type (ii). This will change $\alpha + \delta$, so we can ensure that

$$\delta = -\alpha \tag{4.19}$$

and (4.15) simplifies to

$$\lambda = -\frac{k'}{k}\frac{d}{dk}\left(\frac{k'\alpha}{k^2}\right), \qquad \mu = 0 \tag{4.20}$$

5. SOLUTION OF THE EQUATION

The function G_p depends implicitly on k as well as explicitly on $p=u_p$, so we can write it as $G(u_p, k)$. Then, using (3.2) and (4.20), we can write (4.14) more explicitly as

$$k\frac{\partial}{\partial k}\left[kk'^2\frac{\partial G(u,k)}{\partial k}\right]-k^2G(u,k)+\frac{\partial^2 G(u,k)}{\partial u^2}=\frac{k^2\lambda(k)}{(1-k^2\sin^2 u)^{1/2}} \tag{5.1}$$

(exhibiting also the dependence of λ on k).

From (3.25), (3.44), and (4.13), we have the $k=1$ boundary condition

$$G(u,1)=-\cos u\sum_{j=1}^{N}\frac{N+1-2j}{N^2}\cot\frac{(j-1/2)\pi-u}{N} \tag{5.2}$$

provided $-\pi/2<u<\pi/2$.

We seek to solve these equations for $G(u,k)$ and $\lambda(k)$. From (3.26), (3.36), and (3.44), v_p and x_p are periodic functions of $p=u_p$, of period π, so

$$G(k,u+\pi)=G(k,u) \tag{5.3}$$

Also, v_p and x_p are unchanged by replacing u_p by $u_{sp}=-u_p-\pi$, so we can deduce that

$$G(k,u)=G(k,-u-\pi)=G(k,-u) \tag{5.4}$$

It follows that $G(k,u)$ can be expanded in a Fourier series:

$$G(k,u)=(2/\pi)\left\{\hat{G}_0(k)+2\sum_{n=1}^{\infty}(-1)^n\,\hat{G}_n(k)\cos 2nu\right\} \tag{5.5}$$

Then (5.1) gives

$$k\frac{d}{dk}\left[kk'^2\frac{d}{dk}\hat{G}_n(k)\right]-(4n^2+k^2)\,\hat{G}_n(k)=k^2\lambda(k)\,K_n(k) \tag{5.6}$$

where, for $n\geqslant 0$,

$$\begin{aligned}K_n(k)&=(-1)^n\int_0^{\pi/2}\frac{\cos 2nu\,du}{(1-k^2\sin^2 u)^{1/2}}\\&=\frac{k^{2n}}{2}\frac{\Gamma^2(n+1/2)}{\Gamma(2n+1)}F\left(n+\frac{1}{2},n+\frac{1}{2};2n+1;k^2\right)\end{aligned} \tag{5.7}$$

$F(a,b;c;z)$ is the hypergeometric function. The first (integral) form applies only for n integer; the second is more general and applies for n real, $n > -1/2$.

If $\lambda(k)$ is regarded as given, then (5.6) is an inhomogeneous singular second-order differential linear differential equation for $\hat{G}_n(k)$. The solutions of the homogeneous equation (with $\lambda = 0$) are $K_n(k)$ and $K'_n(k)$, where

$$K'_n(k) = \frac{1}{2} k^{2n} \int_0^\pi \frac{d\theta}{(1 - 2k' \cos\theta + k'^2)^{n+1/2}}$$
$$= \frac{1}{2} \pi k^{2n} F\left(n + \frac{1}{2}, n + \frac{1}{2}; 1; 1 - k^2\right) \tag{5.8}$$

They satisfy the Wronskian relation

$$K'_n \frac{dK_n}{dk} - K_n \frac{dK'_n}{dk} = \frac{\pi}{2k(1-k^2)} \tag{5.9}$$

and K_0, K'_0 are the usual elliptic integrals K. K'.

It follows that the general solution of (5.6) is, for $n \geqslant 0$,

$$\hat{G}_n(k) = \sigma_n K_n(k) + \sigma'_n K'_n(k)$$
$$+ \frac{2}{\pi} \int_k^1 l[K'_n(k)\, K_n(l) - K_n(k)\, K'_n(l)]\, K_n(l)\, \lambda(l)\, dl \tag{5.10}$$

where σ_n, σ'_n are arbitrary parameters, independent of k.

5.1. The Limit $k \to 1$

First consider the low-temperature limit, when $k \to 1$. From (5.2) and (5.5)

$$\lim_{k \to 1} \hat{G}_n(k) = \rho_n \tag{5.11}$$

where

$$\rho_n = \sum_{j=1}^{N-1} \frac{\cot(\pi j/N)}{4j - 2N(2n+1)} \tag{5.12}$$

(If $n = 0$ and N is even, the $j = N/2$ term should be replaced by the limiting value $-\pi/4N$.)

We can in principle make standard low-temperature series expansions about the $k = 1$ case, expanding ψ_{pq} in powers of $k'^2 = 1 - k^2$, provided we restrict attention to the region $-\pi/2 < u_p < u_q < \pi/2$. It follows that g_p, x_p, G_p can be similarly expanded, and hence, from (5.1), so can $\lambda(k)$. Thus, there is at least a formal series expansion:

$$\lambda(k) = \sum_{m=0}^{\infty} \lambda_m (1-k^2)^m \tag{5.13}$$

[Note that the Fourier coefficients $\hat{G}_n(k)$ are *not* series expandable in this form: this is because the higher coefficients in the expansion of $G(k, u)$ are not Fourier analyzable: they have poles at $u = \pm\pi/2$, which in turn is due to the singularity in v when $\sin u = k^{-1}$.]

When $k \to 1$, for $n \geqslant 0$,

$$K_n(k) \sim -\ln k'; \qquad K_n'(k) \to \pi/2 \tag{5.14}$$

and, from (5.13), $\lambda(k)$ tends to finite limit. It follows that the integral in (5.10) is convergent at $l = 1$, and that, for $n \geqslant 0$,

$$\sigma_n = 0, \qquad \sigma_n' = 2\rho_n/\pi \tag{5.15}$$

5.2. The Limit $k \to 0$

Now consider the self-dual case $k \to 0$, when we expect the system to be critical. Then $v_p = 0$, $\theta_p = \phi_p = u_p/N$, and the weights W_{pq}, $\bar{W}_{pq}$ depend on u_p, u_q only via the difference $u_q - u_p$. From (2.13) it follows that g_p/p is a constant (independent of p). From (4.4) and (4.13), $k^2 G_p$, i.e., $k^2 G(u, k)$ must therefore tend to a limit, independent of p or u. Hence $k^2 \hat{G}_n(k)$ must tend to zero for $n \geqslant 1$, while from (4.14) we expect $k^4 \lambda(k)$ to tend to a limit (actually zero).

Now consider Eq. (5.10). When $k \to 0$,

$$\begin{aligned} K_n(k) &\sim \pi 2^{-4n-1} \binom{2n}{n} k^{2n}, && n \geqslant 0 \\ K_n'(k) &\sim n^{-1} 2^{4n-2} \binom{2n}{n}^{-1} k^{-2n}, && n \geqslant 1 \\ K_0'(k) &\sim -\ln k \end{aligned} \tag{5.16}$$

For $n > 0$, we can rewrite the integral of the first term in the integrand of (5.10) as the difference of two integrals: from $l = 0$ to 1, and from $l = 0$ to k. This decomposes $k^2 \hat{G}_n(k)$ into two terms, one of which is proportional to $k^2 K_n'(k)$, and hence does not tend to zero as $k \to 0$, while the other term does tend to zero. It follows that the first term cannot occur, i.e., the net coefficient of $k^2K_n'(k)$ must be zero. Using (5.15), this implies that, for $n \geqslant 1$,

$$\int_0^1 lK_n^2(l)\,\lambda(l)\,dl = -\rho_n \tag{5.17}$$

Equation (5.10), for $n \geqslant 1$, can then be written as

$$\hat{G}_n(k) = -\frac{2}{\pi} K_n'(k) \int_0^k lK_n^2(l)\,\lambda(l)\,dl - \frac{2}{\pi} K_n(k) \int_k^1 lK_n(l)\,K_n'(l)\,\lambda(l)\,dl \tag{5.18}$$

5.3. The Function $\lambda(k)$

Remembering the implicit dependences on k, we have gone from functions of three variables $(\psi_{pq}, \bar{\psi}_{pq})$ to functions of two $(u_p, v_p, x_p, y_p, G_p = G(u_p, k))$, then to functions of one $(\alpha, \beta, \delta, \lambda, \mu)$. We are now at the deepest level of this sequence: we want to solve the linear integral equation (5.17) for $\lambda(k)$.

The following method is probably not the most elegant, but at least it demonstrates that there is a unique solution that is expandable in the form (5.13), and enables us to examine the critical $k \to 0$ behavior.

Define, for m a nonnegative integer and n real, $n > -1/2$:

$$\chi_{mn} = 4 \int_0^1 lK_n^2(l)(1-l^2)^m\,dl \tag{5.19}$$

Then in Appendix A we show that, for $m \geqslant 1$,

$$\begin{aligned} \chi_{0n} &= \sum_{j=0}^{\infty} (j+n+1/2)^{-3} \\ \chi_{1n} &= (2n^2+1/2)\,\chi_{0n} - 1 \\ (m+1)^3\,\chi_{m+1,n} &= (2m+1)(2n^2+m^2+m+1/2)\,\chi_{mn} - m^3\chi_{m-1,n} \end{aligned} \tag{5.20}$$

For n large, χ_{mn} has an asymptotic expansion in powers of n^{-2}:

$$\chi_{mn} = \sum_{r=m+1}^{\infty} c_{mr} n^{-2r} \tag{5.21}$$

with leading coefficient

$$c_{m,m+1} = (m!)^4/[2(2m+1)!] \tag{5.22}$$

Substituting the series expansion (5.13) into (5.17), using (5.14), we get, for $n \geqslant 1$,

$$\sum_{m=0}^{\infty} \chi_{mn} \lambda_m = -4\rho_n \tag{5.23}$$

Equation (5.12) can be written, for $n \geqslant 1$, as

$$\rho_n = \frac{1}{2} \sum_{j=1}^{N-1} \frac{(N-2j)\cot(\pi j/N)}{4N^2 n^2 - (N-2j)^2} \tag{5.24}$$

Clearly, ρ_n can be expanded in powers of n^{-2}. Even though the series (5.21) is only an asymptotic one, it follows that both sides of (5.23) can be expanded in powers of n^{-2}. Equating coefficients of n^{-2} gives a linear equation for λ_0; the coefficient of n^{-4} gives an equation for λ_0 and λ_1; that of n^{-6} gives λ_0, λ_1, λ_2; and so on. In this way we can in principle systematically calculate λ_0, λ_1, λ_2,..., and hence the function $\lambda(k)$. In particular, we get

$$\lambda_0 = N^{-2} \sum_{j=1}^{N-1} (2j - N)\cot(\pi j/N) \tag{5.25}$$

In practice we can streamline this procedure. Consider the equation, true for $n \geqslant 1$,

$$\sum_{m=0}^{\infty} \chi_{mn} \mu_m = \frac{1}{n^2 - a^2} \tag{5.26}$$

where a is fixed, $|a| < 1/2$, and we want to solve for the μ_m. From (5.21), χ_{mn} has an asymptotic expansion in inverse powers of $n^2 - a^2$:

$$\chi_{mn} = \sum_{r=m+1}^{\infty} d_{mr}(n^2 - a^2)^{-r} \tag{5.27}$$

so in terms of these coefficients d_{mr}, (5.26) becomes, for $r \geqslant 1$,

$$\sum_{m=0}^{r-1} d_{mr} \mu_m = \delta_{r,1} \tag{5.28}$$

Also, substituting (5.27) into (5.20), for $m \geqslant 0$ and $r \geqslant 1$,

$$(m+1)^3 d_{m+1,r} = (4m+2) d_{m,r+1} + (2m+1)(2a^2 + m^2 + m + 1/2) d_{mr} - m^3 d_{m-1,r} \tag{5.29}$$

(taking $d_{01} = 1/2$, $d_{-1,r} = 0$, and $d_{mr} = 0$ if $r \leqslant m$).

Multiplying (5.29) by $\mu_m/(2m+1)$, summing over m, and using (5.28), we get, for $r \geqslant 1$,

$$\sum_{m=0}^{r-1} d_{mr} v_m = 0 \tag{5.30}$$

where, for $m \geqslant 0$,

$$v_m = \frac{m^3}{2m-1} \mu_{m-1} - (2a^2 + m^2 + m + 1/2)\, \mu_m + \frac{(m+1)^3}{2m+3} \mu_{m+1} \tag{5.31}$$

(taking μ_{-1} finite).

Now the only solution of (5.30) is

$$v_m = 0, \qquad m \geqslant 0 \tag{5.32}$$

so (5.31) is a recurrence relation for the μ_m. From the $r=1$ case of (5.28) it is readily seen that

$$\mu_0 = 2 \tag{5.33}$$

so the μ_m are defined by (5.31)–(5.33).

However, this recurrence relation is very like that for the χ_{mn}. In fact it becomes precisely the same if in (5.31) we replace μ_m by $(2m+1)\,\chi_{mn}$ and a by n. If we extend the definitions (5.20) to real (positive or negative) values of n, then it follows that

$$\mu_m = (4m+2) \frac{\chi_{m,a} - \chi_{m,-a}}{\chi_{0,a} - \chi_{0,-a}} \tag{5.34}$$

$$= -\frac{4m+2}{\pi^3} \frac{\cos^3 \pi a}{\sin \pi a} (\chi_{m,a} - \chi_{m,-a}) \tag{5.35}$$

From (5.24), $-4\rho_n$ is a linear combination of terms $(n^2 - a^2)^{-1}$, where $a = (N-2j)/2N$. Hence, λ_m is the same linear combination of the μ_m obtained above:

$$\lambda_m = \frac{2m+1}{\pi^3 N^2} \sum_{j=1}^{N-1} (N-2j) \left\{ \sin^2 \frac{\pi j}{N} \right\} (\chi_{m,(N-2j)/2N} - \chi_{m,(2j-N)/2N}) \tag{5.36}$$

Using (5.13) and (5.19), we get

$$\lambda(k) = \frac{4}{\pi^3 N^2} \sum_{j=1}^{N-1} (N-2j) \sin^2 \frac{\pi j}{N}$$
$$\times \int_0^1 \frac{2 - k^2 - l^2 + k^2 l^2}{(k^2 + l^2 - k^2 l^2)^2} [K^2_{(N-2j)/2N}(l) - K^2_{(2j-N)/2N}(l)]\, l\, dl \tag{5.37}$$

The problem is now solved: $\lambda(k)$ is given by (5.37); $\hat{G}_n(k)$ by (5.10), (5.15), (5.18); $G_p = G(k, u_p)$ by (5.5); and x_p by (4.13). Then y_p is given, to within an additive multiple of k^2/k'^2, by (4.12); Λ_{pq}, ψ_{pq} are given by (3.42)–(3.46), to within additive functions of $u_q - u_p$ only—these can be determined from (3.25).

The resulting expressions can probably be simplified: some thoughts in this direction are offered in Appendix B.

5.4. Ising Case ($N=2$)

When $N=2$ the chiral Potts model reduces to the usual Ising model. In this case the above equations give

$$
\begin{aligned}
\rho_n &= (-\pi/8)\,\delta_{n,0}, \qquad \lambda(k) = 0 \\
\hat{G}_n(k) &= (-1/4)\,K_0'(k)\,\delta_{n,0} \\
G(k, u) &= -(2\pi)^{-1}\,K_0'(k) \\
x_p &= -k^2 K_0'(k)/(2\pi \cos v_p) \\
y_p &= \frac{k}{4\pi k'^2}\left\{\frac{\pi w_p}{2K_0(k)} + K_0'(k)\,\frac{d}{dw_p}\ln \Theta_1(w_p, k)\right\}
\end{aligned}
\tag{5.38}
$$

where $\Theta_1(u, k)$ is the usual Jacobi elliptic theta function, and w_p is given by

$$
\sin u_p = \mathrm{sn}(w_p, k), \qquad \cos u_p = \mathrm{cn}(w_p, k), \qquad \cos v_p = \mathrm{dn}(w_p, k) \tag{5.39}
$$

I have verified that these results follow from Onsager's solution,[1] using Eq. (11.7.24) of ref. 2.

This w_p is the "natural" rapidity variable, in the sense that the Boltzmann weights and free energies (for $N=2$) are functions only of k and the difference $w_q - w_p$. From (3.44), $g_p = -k^2 K_0'(k) w_p/2\pi$, so g_p is proportional to this variable. Indeed, for any model with the difference property, this is a consequence of (2.13). This suggests that in some sense g_p may always be a "natural" rapidity variable: it would be interesting to consider (for arbitrary N) the analyticity properties of ψ_{pq} as a function of q_p and g_q for fixed k.

6. CRITICAL BEHAVIOR

When $k \to 0$, the integral in (5.37) is dominated by the contribution from l small, and the sum by the $j=1$ and $N-1$ terms. Then, using the second form in (5.7),

$$
\lambda(k) \sim -\frac{8r\sin(2\pi/N)}{\pi N}\int_0^1 \frac{l^{\theta-1}\,dl}{(k^2+l^2)^2} \tag{6.1}
$$

where

$$r=[(N-2)/4\pi^2 N]\tan(\pi/N)\,\Gamma^4(1/N)/\Gamma^2(2/N) \tag{6.2}$$

$$\theta=4/N \tag{6.3}$$

The integral in (5.38) can be extended to the interval $0<l<\infty$. Then from Eq. (3.194.6) of Ref. 13 we obtain

$$\lambda(k)\sim -4N^{-2}(N-2)\,rk^{\theta-4} \tag{6.4}$$

Note that $k^4\lambda(k)$ does tend to a limit (namely zero) as $k\to 0$, as expected.

From (5.10) and (5.18), it follows that for $n\geqslant 0$, as $k\to 0$,

$$\hat{G}_n(k)\sim\frac{\pi r 2^{-4n-1}}{(2Nn-N+2)}\binom{2n}{n}k^{\theta+2n-2} \tag{6.5}$$

These results are true for $N\geqslant 3$. When $N=2$, $r=0$, and (6.5) is indeterminate for $n=0$: from (5.38) and (5.16) we then have $\hat{G}_0(k)\sim(\ln k)/4$.

Using (4.13) and (5.5), working to order k^θ in terms that are independent of u_p, and to order $k^{\theta+2}$ in terms that do depend on u_p:

$$x_p=-\frac{r}{N-2}k^\theta+\frac{r}{N^2-4}k^{\theta+2}\cos 2u_p \tag{6.6}$$

From (4.20), working to order k^2:

$$\alpha(k)\sim(-2r/N)\,k^\theta+\alpha_1 k^2 \tag{6.7}$$

where α_1 is a constant of integration. It follows that Eqs. (4.12) are satisfied (to the relevant orders), provided y_p is given to order $k^{\theta+2}$ by

$$y_p=k^2(\varepsilon+\alpha_1 u_p)-\frac{r}{N^2-4}k^{\theta+2}\sin 2u_p \tag{6.8}$$

where ε is another constant (independent of k and p).

The constant ε arises from the degree of freedom discussed in (4.16): it cancels out of (3.12), (3.16), (3.45), and (3.46), and so is of no interest. The constant α_1 can be determined from the rotation symmetry requirements (3.32) and (3.36). Using (3.44), they imply the exact relation

$$y_{Rp}-y_p=(N-1)k^2/(Nk'^2) \tag{6.9}$$

Since y_{Rp} is obtained from y_p by replacing u_p by $u_p+\pi$, we see that (6.8) is consistent with (6.9) provided

$$\alpha_1=(N-1)/N\pi \tag{6.10}$$

Substituting these results for x_p, y_p into (3.45) and (3.46) and integrating, we get (to order $k^{\theta+2}$)

$$\Lambda_{pq} = J(u_q - u_p) + \frac{N-1}{2N\pi} k^2(u_q - u_p)\cos(u_p + u_q) - \frac{Nrk^{\theta+2}}{(N+2)(N^2-4)} \sin(u_q - u_p) \tag{6.11}$$

The function $J(u_q - u_p)$ is independent of k, being the value of Λ_{pq} when $k=0$, i.e., at criticality. Our model then reduces to the self-dual Z_N model of Fateev and Zamolodchikov,[15,16] and we regain the "difference property" of previously solved models: the weights and free energies depend on u_p and u_q only via $u_q - u_p$. (There are other models,[17–19] distinct from ours and having the difference property that also become the Fateev–Zamolodchikov model at criticality.)

Fateev and Zamolodchikov obtained $J(u)$ by the "inversion relation" method. In principle it should be possible to obtain it more rigorously from the above equations by integrating (3.45) all the way from $k=1$ [where (3.25) and (3.42) give Λ_{pq}] down to $k=0$. I have not yet done this, so here I simply follow Fateev and Zamolodchikov. The method depends on the difference property: in particular, from (3.22) we have $f_{pq} = f(u_q - u_p)$, where

$$\ln f(u) = \tfrac{1}{2}\ln N - [(N-1)/N]\ln[2\sin(u/2)] + N^{-1}\sum_{j=1}^{N-1} j\ln[2\sin(u+\pi j)/N] \tag{6.12}$$

The inversion relation (3.40) implies

$$J(u) + J(-u) = 0 \tag{6.13}$$

while the rotation symmetry (3.30) implies

$$J(u) - \ln f(u) = J(\pi - u) - \ln f(\pi - u) \tag{6.14}$$

If we make the standard assumption[12] that ψ_{pq}, $\bar{\psi}_{pq}$, Λ_{pq} are analytic in the vertical strip $0 \leqslant \mathrm{Re}(u_q - u_p) \leqslant \pi$, apart from possible logarithmic branch point singularities at $u_q - u_p = 0$, π arising from the Boltzmann weights becoming infinite, then $J(u)$ is analytic for $0 \leqslant \mathrm{Re}(u) < \pi$, and it follows that

$$J(u) = \ln f(u) - \ln[W_{pq}(1,1)\,\bar{W}_{pq}(1,1)] - \int_0^\infty \frac{\mathrm{sh}\,ux\,\mathrm{sh}(\pi-u)x\,\mathrm{sh}(N-1)\,\pi x}{x\,\mathrm{ch}^2\,\pi x\,\mathrm{ch}\,N\pi x}\,dx \tag{6.15}$$

where $W_{pq}(1, 1)$, $\bar{W}_{pq}(1, 1)$ are here evaluated from (3.1)–(3.8) with $k=0$ (and hence $\theta_p = \phi_p = u_p/N$), and with $u_q - u_p = u$. (This u is the α of Ref. 15.) (The derivation parallels that of Section 5.1 of ref. 12. Note that I am using this nonrigorous "inversion relation method" only as a shortcut to calculating $J(u)$. It is *not* the method used elsewhere in this paper.)

The Boltzmann weights W_{pq}, $\bar{W}_{pq}$ can be expanded in powers of k^2, so k^2 plays the role in this model of the temperature deviation from criticality, $T - T_c$. We note immediately from (6.11) that the dominant singular contribution to the free energy is proportional to $k^{\theta+2}$. If we make the standard convention that this is $(T_c - T)^{2-\alpha}$, this α being the specific heat critical exponent, then we have

$$\alpha = 1 - \theta/2 = 1 - 2/N \tag{6.16}$$

which is the result (1.1).

7. SUMMARY

Apart from the number N of spin states, there are three parameters in the chiral Potts model; p, q, and k. In terms of p and k (we often let the k dependence be implicit) one can define related variables u_p, v_p, θ_p, ϕ_p, as in (3.1)–(3.3).

The Boltzmann weight function W_{pq} depends on these parameters only via two quantities: $\theta_q - \phi_p$ and $\theta_p - \phi_q$. Similarly, $\bar{W}_{pq}$ depends only on $\theta_q - \theta_p$ and $\phi_q - \phi_p$. I have shown in this paper how these simple properties, together with the star-triangle relation satisfied by the model, can be used to calculate the free energy. The solution is contained in Eq. (5.37) for $\lambda(k)$; (5.10) or (5.18) for $\hat{G}_n(k)$; (5.5) for $G_p = G(u_p, k)$; (4.13) for x_p; (4.12) for y_p; (3.45) and (3.46) for Λ_{pq}; (3.44) for g_p, h_p; (3.41) and (3.42) for the free energies $\psi_{pq}^{(\mathrm{Sq})}$, ψ_{pq}, $\bar{\psi}_{pq}$. The partition functions of the honeycomb, triangular, and square lattices are given by (2.11) and (2.16). In particular, in Section 6 these equations are used to obtain the critical behavior of the free energy, the value $1 - 2/N$ is found for the exponent α.

A key equation is (5.1). The homogeneous form of this equation (with $\lambda = 0$) is a linear partial differential equation for $G(u, k)$. This equation separates, having solutions $K_n(k) \cos 2nu$ and $K_n'(k) \cos 2nu$. This has been an essential feature of this solution.

More remains to be done: one would hope that the result (5.37) for $\lambda(k)$ and the consequent results for the other functions can be simplified. In Appendix B a few thoughts in this direction are put forward.

This lengthy calculation has been motivated by the fact that the chiral Potts model appears to provide a quite new solution of the star-triangle or Yang–Baxter relation: one in which the usual difference property

(W_{pq}, $\bar{W}_{pq}$ depending only on $p-q$ and k) is absent. In previously solved planar models, notably the Ising,[1] eight-vertex,[10] and hard-hexagon[14] models, this difference property has led naturally to the use of Jacobi elliptic functions and to an interesting collaboration between physics and mathematics. It seems likely that this new solution (which includes the Ising model as a special case) will extend this interaction.

APPENDIX A

Here I establish the relations (5.20) for the integrals (5.19). I generalize the problem and define, for m a nonnegative integer and $-1<c-a-b<1$, $c>0$,

$$I_m(a, b, c)=\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c-a)\,\Gamma(c-b)}{\Gamma^2(c)}\,\frac{\sin \pi(a+b-c)}{\pi}$$

$$\times \int_0^1 (1-x)^m\, x^{c-1} F(a, b; c; x)\, F(c-a, c-b; c; x)\, dx \qquad \text{(A1)}$$

$F(a, b; c; x)$ is the hypergeometric function (Section 9.100 of ref. 13),

$$F(a, b; c; x)=\frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\,\Gamma(b+j)\,x^j}{\Gamma(c+j)\,j!} \qquad \text{(A2)}$$

Also, from §9.131.2 of ref. 13:

$$F(c-a, c-b; c; x)$$

$$=\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\, F(c-a, c-b; c-a-b+1; 1-x)$$

$$+(1-x)^{a+b-c}\,\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\, F(a, b; a+b-c+1; 1-x) \qquad \text{(A3)}$$

Now use the general formula (A2) to expand the rhs of (A3) in powers of $1-x$. Substitute the result, for $F(c-a, c-b; c; x)$ only, into (A1). We get a weighted sum of integrals

$$\int_0^1 (1-x)^{\rho-1}\, x^{c-1} F(a, b; c; x)\, dx \qquad \text{(A4)}$$

where $\rho>0$, $c+\rho-a-b>0$. From §7.512.4 of ref. 13, (A4) is equal to

$$\frac{\Gamma(c)\,\Gamma(\rho)\,\Gamma(c+\rho-a-b)}{\Gamma(c+\rho-a)\,\Gamma(c+\rho-b)} \qquad \text{(A5)}$$

Using also the identity $\Gamma(1-p)\,\Gamma(p)=\pi \operatorname{cosec} \pi p$, we thus obtain

$$I_m(a, b, c)=\sum_{j=0}^{\infty} \frac{(j+m)!}{j!}$$
$$\times \left\{\frac{\Gamma(c-a+j)\,\Gamma(c-b+j)\,\Gamma(c-a-b+j+m+1)}{\Gamma(c-a+j+m+1)\,\Gamma(c-b+j+m+1)\,\Gamma(c-a-b+j+1)}\right.$$
$$\left.-\frac{\Gamma(a+j)\,\Gamma(b+j)\,\Gamma(a+b-c+j+m+1)}{\Gamma(a+j+m+1)\,\Gamma(b+j+m+1)\,\Gamma(a+b-c+j+1)}\right\} \qquad \text{(A6)}$$

In particular, for $m=0$,

$$I_0(a, b, c)=\sum_{j=0}^{\infty}\left\{\frac{1}{(c-a+j)(c-b+j)}-\frac{1}{(a+j)(b+j)}\right\} \qquad \text{(A7)}$$

For $m=1, 2, 3,\ldots$, the summand in (A6) is still a rational function of j, though it becomes progressively more complicated. Making a partial fraction decomposition of the summand, $I_m(a, b, c)$ can in principle be expressed as a weighted sum of Euler psi functions (using §8.363.3 of Ref. 13). Proceeding in this way, we have obtained the relations

$$I_1(a, b, c)=\frac{(1-a-b)c+a^2+b^2-1}{(a-b)^2-1}\,I_0(a, b, c)+\frac{2(a+b-c)}{(a-b)^2-1} \qquad \text{(A8)}$$

$$I_{m+1}(a, b, c)=\frac{2m+1}{m+1}\,\frac{(1-a-b)c+a^2+b^2-1-m-m^2}{(a-b)^2-(m+1)^2}\,I_m(a, b, c)$$
$$+\frac{m[m^2-(c-a-b)^2]}{(m+1)[(a-b)^2-(m+1)^2]}\,I_{m-1}(a, b, c) \qquad \text{(A9)}$$

From (5.7), (5.19), and (A1),

$$\chi_{mn}=\frac{\pi}{2\sin \pi(a+b-c)}\,I_m(a, b, c) \qquad \text{(A10)}$$

evaluated in the limit when a, b, and c become $n+1/2$, $n+1/2$, and $2n+1$, respectively. Dividing (A7)–(A9) by $2(a+b-c)$ and taking this limit, we obtain the desired formulas (5.20).

APPENDIX B

Equation (5.1), or equivalently (4.14), is a linear partial differential equation for $G(u, k)$. It is a corollary, via (4.13), of the pair of (4.12) of pde's for x_p, y_p. Once these functions are known, g_p, h_p, Λ_{pq}, ψ_{pq}, and $\bar{\psi}_{pq}$ are given by (3.41)–(3.46), to within appropriate integration "constants."

All these equations are linear. It is natural to regard the right-hand sides of (5.1), (4.14), and (4.12), involving α, β, δ, and λ, as "forcing terms," and to start by considering the homogeneous equations obtained by replacing α, β, δ, and λ by zero.

Here I show that these homogeneous equations have a simple set of solutions. In the course of deriving (5.1), I noted that the homogeneous equation has the solution $G(u, k) = \sec v = (1 - k^2 \cos^2 u)^{-1/2}$. However, the lhs of (5.1) is translation-invariant under $u \to u - \xi$, where ξ is a constant (independent of both u and k). Defining

$$u' = u - \xi, \qquad \sin v' = k \sin u', \qquad \cos v' = (1 - k^2 \sin^2 u')^{1/2} \tag{B1}$$

(so v' is a function of u and k), it follows that a solution of the homogeneous equation (5.1) is

$$G(u, k) = \sin \xi / \cos v' \tag{B2}$$

Writing u'_p, v'_p for the values of u', v' when u is replaced by u_p, we immediately obtain from (4.13)

$$x_p = k^2 \sin \xi / (\cos v_p \cos v'_p) \tag{B3}$$

Then both the homogeneous equations (4.12) are satisfied by

$$y_p = k^2 (\cos u_p \cos u'_p + k'^2 \sin u_p \sin u'_p)/(k'^2 \cos v_p \cos v'_p) \tag{B4}$$

and (3.45) and (3.46) by

$$\Lambda_{pq} = \ln \left\{ \frac{\sin u'_q \cos v_p + \sin u_p \cos v'_q}{\sin u'_p \cos v_q + \sin u_q \cos v'_p} \times \frac{\cos u'_q \cos v_p + \cos u_p \cos v'_q}{\cos u'_p \cos v_q + \cos u_q \cos v'_p} \right\} \tag{B5}$$

The equations are linear, so any linear combination of the forms (B3)–(B5) (with different values of ξ) is also a solution. For the $N = 2$ Ising case, α, β, δ, and λ are indeed zero, and the result (5.39) for x_p can be written as

$$x_p = -\frac{k^2}{4\pi \cos v_p} \int_{-\infty}^{\infty} \frac{d\alpha}{(1 + k^2 \sinh^2 \alpha)^{1/2}} \tag{B6}$$

For a given value of u between $-\pi/2$ and $\pi/2$, change the contour of integration to the line $I_m(\alpha) = u$; then change the variable to ξ, where $\alpha = i(u - \xi)$; so ξ is pure imaginary. We get

$$x_p = (i/4\pi) \int_{-i\infty}^{i\infty} [k^2/(\cos v_p \cos v'_p)]\, d\xi \tag{B7}$$

It follows at once that the Ising model result for Λ_{pq} is obtained from (B5) by multiplying by $i/4\pi \sin \xi$ and integrating with respect to ξ from $-i\infty$ to $i\infty$. [Except for an additive term that depends on k, u_p, and u_q only via $u_q - u_p$: we have noted before that such terms cancel out of (3.45) and (3.46), but can be determined from the low-temperature $k=1$ limit.]

It seems likely that the general homogeneous solution of (5.1), (4.12), (3.45), and (3.46) can be expressed as a linear combination of (B3)–(B5).

For $N \geqslant 3$, the functions α, β, δ, and λ are not all zero. Even so, it may be possible to extend these ideas to the inhomogeneous equations, and thereby obtain a more transparent and explicit solution for the free energy of the chiral Potts model.

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